Mathematical Association of NSW

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MASTERING HSC MATHEMATICS

YEAR 12 EXTENSION 2 MATHEMATICS

NEW STAGE 6 HSC SYLLABUS

FOR STUDENTS AND TEACHERS

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Exercise 1B

Direct Proof

Fundamentals

Fundamentals 1
Describe the general steps to prove that an integer n is

(a) even. (b) odd. (c) divisible by a.

Question 1 Prove that

(a) if m and n are even, then mn is even.
(b) if m and n are odd, then mn is odd.
(c) if m and n are even, then $m + n$ is even.
(d) if m and n are odd, then $m + n$ is even.

Question 2 Prove that

(a) if n is even, then n^2 is even.
(b) if n is even, then $n^2 + 1$ is odd.
(c) if n is odd, then n^2 is odd.
(d) if n is odd, then $n^2 + 1$ is even.

Question 3 Prove that if m is even and n is odd, then

(a) $m + n$ is odd.
(b) $m - n$ is odd.

Question 4

(a) Prove that $n^2 + n$ is always even.
(b) Prove that $n^3 - n$ is divisible by 6.
(c) Prove that n^2 is even if and only if n is even.
(d) Prove that n is divisible by 6 if and only if n is divisible by 2 and 3.

Question 5 Let n be odd. Prove that n^2 has a remainder of 1 when divided by 8.

Question 6 Prove that the sum of three consecutive integers is always divisible by 3.

Question 7 Prove that the sum of two consecutive positive powers of 4 is always a multiple of 20.

Question 8 Prove that $\frac{n}{3} + \frac{n^2}{2} + \frac{n^3}{6}$ is an integer $\forall n \in \mathbb{Z}$.

Question 9 Prove that every odd integer is the sum of two consecutive integers.

Question 10 [Application of binomial expansions]

Prove that $9^n - 1$ is divisible by 8.

Question 11 Prove that the difference between the squares of any two consecutive odd integers is always divisible by 8.

Question 12 Prove that if the sum of the digits of a 3-digit number is divisible by 3, then the number itself is divisible by 3.

Question 13 Prove that every odd integer is the difference between two consecutive perfect squares.

Challenge Problems

Problem 1 [Modular arithmetic properties]

(a) Prove that if a has a remainder of b when it is divided by n, then a^2 and b^2 will have the same remainder when they are divided by n.

(b) Prove that if a has a remainder of b when it is divided by n, then ac and bc will have the same remainder when they are divided by n.

Problem 2 Prove that $\forall n \in \mathbb{Z}^+$, $n \geq 3$, $\exists p$ prime such that $n < p < n!$

Problem 3 Prove that a number is divisible by 8 if and only if the last three digits themselves form a number that is divisible by 8.

Problem 4 Let p be a prime number and let q be some positive integer. Find the smallest value of q such that $p + q$ is never prime.

Problem 5 [Application of the Sophie Germain Identity]

(a) Show that $a^4 + 4b^4 = (a^2 + 2ab + 2b^2)(a^2 - 2ab + 2b^2)$.

(b) Hence, show that if $n > 1$, then $n^4 + 4^n$ is composite.

Problem 6 [Trivial proof]

Prove that no three positive integers a, b and c satisfy the equation

$$a^n + b^n = c^n$$

for any integer value of $n \geq 3$.

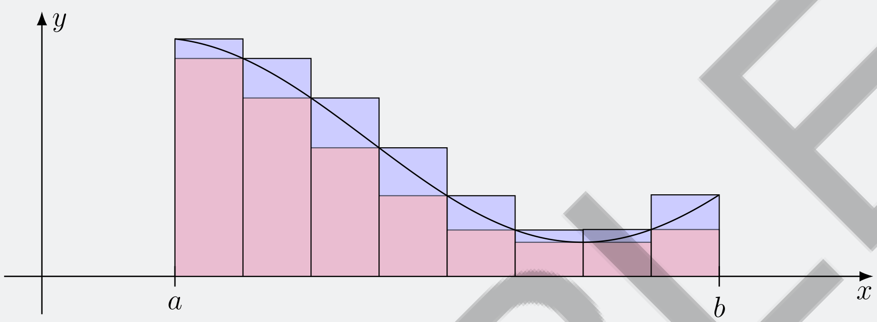
Exercise 1I

Inequalities using Integration

Fundamentals

Fundamentals 1

The diagram below shows a function $f(x)$ and a number of upper and lower-bound rectangles.

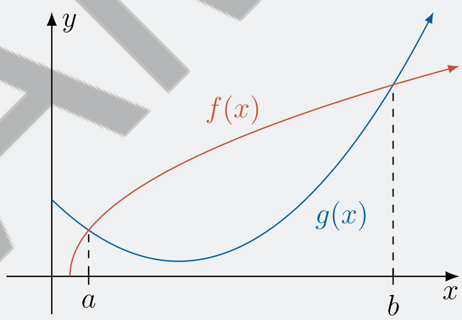


Let the total area of the upper and lower-bound rectangles be U and L respectively.

$$\_\_ < \int_a^b f(x)\,dx < \_\_$$

Fundamentals 2

The diagram below shows some function $f(x) \geq g(x)$ for $x \in [a, b]$.

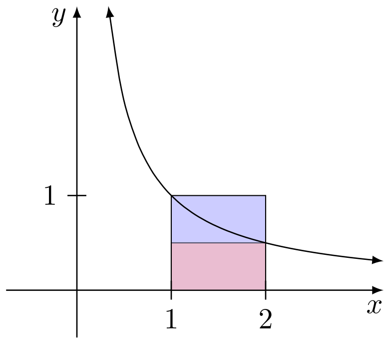


It follows that

$$\int_a^b f(x)\,dx > \int_?^? \_\_\_\_\_\ dx$$

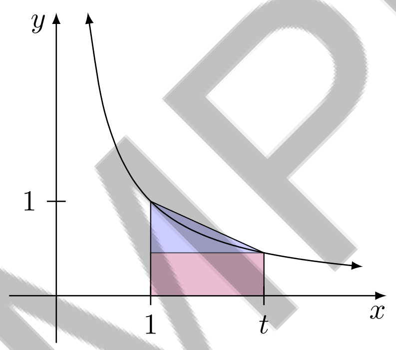
Equality is lost because although the functions were equal to each other originally at the point of intersections, their areas are often not equal and so their integrals are not necessarily equal.

Question 1 The diagram below shows a section of the graph of $y = \frac{1}{x}$. Consider the region $x \in [1, 2]$.



Use the diagram to prove that $\frac{1}{2} < \ln 2 < 1$.

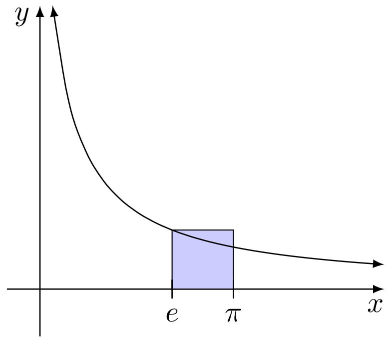
Question 2 The diagram below shows a section of the graph of $y = \frac{1}{x}$. Consider the region $x \in [1, t]$.



Use the diagram to prove that

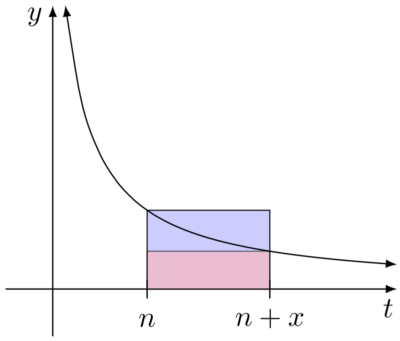
$$1 - \frac{1}{t} \leq \ln t \leq \frac{1}{2}\left(t - \frac{1}{t}\right).$$

Question 3 The diagram below shows a section of the graph of $y = \frac{1}{x}$. Consider the upper-bound rectangle in the domain $x \in [e, \pi]$.



Use the diagram to show that $e^\pi > \pi^e$.

Question 4 The diagram below shows a section of the graph of $y = \frac{1}{t}$.



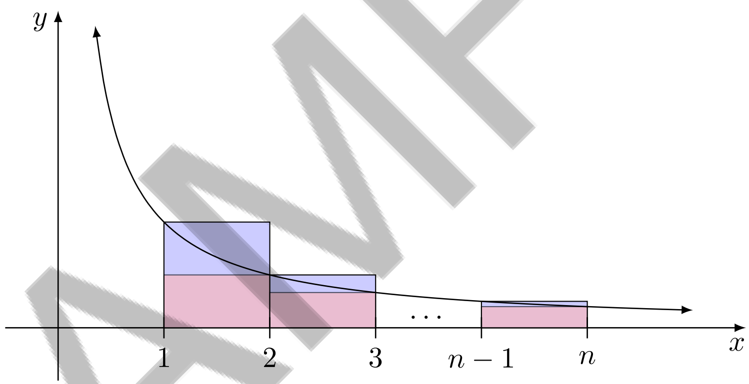
Consider the region $t \in [n, n+x]$, where $x > n$.

(a) Prove that $\dfrac{x}{1+\frac{x}{n}} < n \ln\left(1+\dfrac{x}{n}\right) < x$.

(b) Hence, show that $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$.

Question 5 [Harmonic Series]

The diagram below shows the graph of $y = \frac{1}{x}$. Upper and lower-bound rectangles of unit width are constructed over the domain $x \in [1, n]$.



Define the series $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

(a) Show that

$$\frac{1}{n} + \ln n < H_n < 1 + \ln n.$$

(b) Hence, find two integers which are lower and upper bounds of the following sum.

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2020}$$

(c) Does the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$$

have a finite limit?

Exercise 2G

Applications of de Moivre's Theorem

Fundamentals

Fundamentals 1

Let α, β and γ be the roots of $ax^3 + bx^2 + cx + d = 0$. Write the following in terms of a, b, c, d.

(a) $\alpha + \beta + \gamma$ (b) $\alpha\beta + \beta\gamma + \alpha\gamma$ (c) $\alpha\beta\gamma$

Fundamentals 2

To express $\cos(n\theta)$ as a polynomial in terms of $\cos\theta$, follow the following steps.

(a) Define $z =$ ________ $+ i$ ________.

(b) Simplify z^n using ________________ theorem.

(c) Expand z^n manually using P______ triangle, or using the b________ expansion.

(d) Equate the real/imaginary (circle one) components of the two expressions for z^n.

(e) Turn all even powers of $\sin\theta$ into powers of ________ using the identity ______________.

Fundamentals 3

To express $\tan(n\theta)$ as a rational expression in terms of $\tan\theta$, follow the following steps.

(a) Obtain expressions for $\cos n\theta$ and ______ using the same steps outlined above. Keep the expressions as they are after equating real/imaginary components and do not modify them.

(b) D________ the two expressions to obtain $\tan n\theta$ in terms of powers of $\cos\theta$ and $\sin\theta$.

(c) Divide the top and bottom by the highest power cosine/sine (circle one) term.

(d) Simplify and express everything in terms of ________

Question 1 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$

(b) $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$

(c) $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

(d) $\cot 3\theta = \dfrac{3\cot^2\theta - 1}{\cot^3\theta - 3\cot\theta}$

Question 2 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

(b) $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

Question 3 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(b) $\sin 4\theta = 4\sin\theta\cos^3\theta - 4\sin^3\theta\cos\theta$

(c) $\tan 4\theta = \dfrac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta}$

(d) $\cot 4\theta = \dfrac{4\cot^3\theta - 4\cot\theta}{\cot^4\theta - 6\cot^2\theta + 1}$

Question 4 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 5\theta = \cos^5\theta - 10\sin^2\theta\cos^3\theta + 5\sin^4\theta\cos\theta$

(b) $\sin 5\theta = \sin^5\theta + 5\sin\theta\cos^4\theta - 10\sin^3\theta\cos^2\theta$

Question 5 Let $z = \cos\theta + i\sin\theta$. Prove the following trigonometric identities.

(a) $\cos 6\theta = \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta$

(b) $\sin 6\theta = 6\sin\theta\cos^5\theta - 20\sin^3 x\cos^3\theta + 6\sin^5\theta\cos\theta$

Question 6 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$ for $n \in \mathbb{Z}^+$.

(b) Hence, show that $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$.

(c) Calculate $\int_0^{\frac{\pi}{2}} \cos^5\theta\, d\theta$.

Question 7

(a) Use a similar technique to the previous question to prove that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3).$$

(b) By finding a suitable substitution for θ, deduce that

$$\sin^4\theta = \frac{1}{8}(\cos 4\theta - 4\cos 2\theta + 3).$$

(c) Hence, show that $\cos^4\theta + \sin^4\theta = \frac{1}{4}(\cos 4\theta + 3)$.

Question 8

(a) Show that

$$\cos^6\theta = \frac{1}{32}(10 + 15\cos 2\theta + 6\cos 4\theta + \cos 6\theta).$$

(b) Find a similar result for $\sin^6\theta$.

Question 9 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^n + z^{-n} = 2\cos n\theta$.

(b) Hence, solve $z^4 + 4z^3 + 2z^2 + 4z + 1 = 0$.

Question 10 [Guided question for a classic problem]

Define the following cubic polynomial.

$$P(x) = 8x^3 - 6x - 1$$

You may assume that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(a) Let $x = \cos\theta$. Show that solving $P(x) = 0$ is equivalent to solving $\cos 3\theta = \dfrac{1}{2}$.

(b) Solve the trigonometric equation to find 3 distinct values of θ.

(c) Hence, write down the three zeroes of $P(x)$.

(d) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right) + \cos\left(\frac{5\pi}{9}\right) + \cos\left(\frac{7\pi}{9}\right).$$

(e) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right) + \cos\left(\frac{\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right) + \cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right).$$

(f) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{5\pi}{9}\right)\cos\left(\frac{7\pi}{9}\right).$$

Question 11 You may assume that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

(a) Find the zeroes of $P(x) = 8x^3 - 6x + 1$.

(b) Show that

$$\cos\left(\frac{2\pi}{9}\right) + \cos\left(\frac{4\pi}{9}\right) = \cos\left(\frac{\pi}{9}\right).$$

(c) Find the exact value of

$$\cos\left(\frac{\pi}{9}\right)\cos\left(\frac{2\pi}{9}\right)\cos\left(\frac{4\pi}{9}\right).$$

Question 12

(a) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

(b) Hence, solve $8x^4 - 8x^2 + 1 = 0$.

(c) Hence, find the exact values of $\cos\left(\dfrac{\pi}{8}\right)$ and $\cos\left(\dfrac{5\pi}{8}\right)$.

Question 13 You may assume that $\tan 3\theta = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

(a) Solve the polynomial equation $t^3 - 3t^2 - 3t + 1 = 0$.

(b) Find the exact value of $\tan\left(\dfrac{\pi}{12}\right)$ and $\tan\left(\dfrac{5\pi}{12}\right)$.

Question 14

(a) Prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

(b) Hence, solve the polynomial equation $16x^4 - 20x^2 + 5 = 0$.

(c) Hence, show that

$$\cos\left(\frac{\pi}{10}\right) = \frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}.$$

(d) Write down the exact value of $\cos\left(\dfrac{3\pi}{10}\right)$.

Question 15

(a) Prove that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$.

(b) Show that $x = \sin\left(\dfrac{\pi}{10}\right)$ is a solution to the polynomial equation $16x^5 - 20x^3 + 5x - 1 = 0$.

(c) Find the polynomial $P(x)$ such that $(x-1)P(x) = 16x^5 - 20x^3 + 5x - 1$.

(d) Find the value of a such that $P(x) = (4x^2 + ax - 1)^2$.

(e) Hence, find an exact value for $\sin\left(\dfrac{\pi}{10}\right)$.

Question 16

(a) Show that $\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$.

(b) Hence, find all the roots of the polynomial $32x^6 - 48x^4 + 18x^2 - 1 = 0$.

(c) Show that

$$\cos\left(\frac{\pi}{12}\right)\cos\left(\frac{5\pi}{12}\right) = \frac{1}{4}.$$

(d) Find the exact value of

$$\cos^2\left(\frac{\pi}{12}\right) + \cos^2\left(\frac{5\pi}{12}\right).$$

(e) Hence, show that

$$\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{5\pi}{12}\right) = \frac{\sqrt{6}}{2}.$$

Question 17

(a) Show that

$$\cot 6\theta = \frac{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}.$$

(b) Find the exact value of $\tan\left(\frac{\pi}{12}\right)\tan\left(\frac{5\pi}{12}\right)$.

(c) Find the exact value of $\tan^2\left(\frac{\pi}{12}\right) + \tan^2\left(\frac{5\pi}{12}\right)$.

(d) Hence, find $\tan\left(\frac{\pi}{12}\right) + \tan\left(\frac{5\pi}{12}\right)$.

(e) Write down the equation of the quadratic polynomial that has roots $\tan\left(\frac{\pi}{12}\right)$ and $\tan\left(\frac{5\pi}{12}\right)$.

(f) Hence, find the exact value of $\tan\left(\frac{\pi}{12}\right)$ and $\tan\left(\frac{5\pi}{12}\right)$.

Question 18

(a) Prove that

$$\tan 5\theta = \frac{\tan^5\theta - 10\tan^3\theta + 5\tan\theta}{5\tan^4\theta - 10\tan^2\theta + 1}.$$

(b) Hence, find the roots of the polynomial $t^4 - 10t^2 + 5 = 0$.

(c) Show that $\tan\left(\frac{\pi}{5}\right)\tan\left(\frac{2\pi}{5}\right) = \sqrt{5}$.

(d) Show that $\tan\left(\frac{\pi}{5}\right) + \tan\left(\frac{2\pi}{5}\right) = \sqrt{10 + 2\sqrt{5}}$.

(e) Hence, prove that $\tan\left(\frac{\pi}{5}\right) = \sqrt{5 - 2\sqrt{5}}$.

(f) Find the exact value of $\tan\left(\frac{2\pi}{5}\right)$ and justify your answer.

Challenge Problems

Problem 1 Show that if $-1 < r < 1$, then

$$1 + r\cos\theta + r^2\cos 2\theta + r^3\cos 3\theta + \cdots = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}.$$

Problem 2 Let $z = \cos\theta + i\sin\theta$.

(a) Show that $z^k + z^{-k} = 2\cos k\theta$, where $k \in \mathbb{Z}^+$.

(b) Let $n \in \mathbb{Z}^+$. Prove the following identity.

$$(2\cos\theta)^{2n} = 2\sum_{k=0}^{n}\binom{2n}{k}\cos(2n-2k)\theta.$$

(c) Hence, prove that

$$\int_0^{\frac{\pi}{2}} \cos^{2n}\theta \, d\theta = \frac{\pi}{2^{2n+1}}\binom{2n}{n}.$$

Problem 3 Let $z = \cos\theta + i\sin\theta$.

(a) Simplify $\left(z + \frac{1}{z}\right)^n z^n$.

(b) Show that

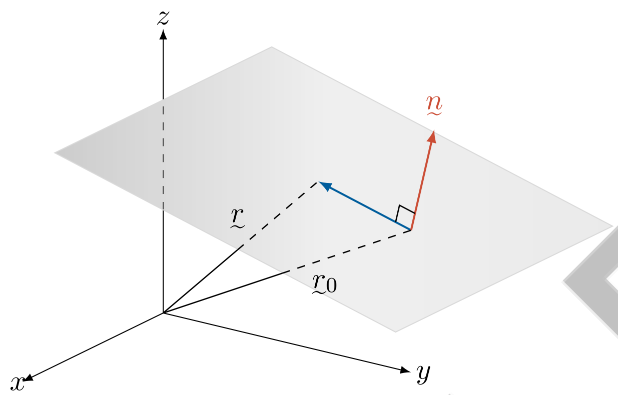
$$2^n\cos^n\theta\cos(n\theta) = \sum_{k=0}^{n}\binom{n}{k}\cos(2k\theta).$$

(c) Hence, show that

$$\int_{-\pi}^{\pi} \cos^n\theta\cos n\theta \, d\theta = \frac{\pi}{2^{n-1}}.$$

Question 12 [Equation of a plane]

A plane in 3D space can be defined by a point on the plane and a vector perpendicular to the plane called the normal vector. This works similarly to how a line in 2D space can be defined by a point on the line and a gradient.



The diagram above shows two vectors $\underset{\sim}{r}$ and $\underset{\sim}{r}_0$ representing the position vectors of points $P(x, y, z)$ and $P_0(x_0, y_0, z_0)$ on a plane. Let $\underset{\sim}{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be the vector perpendicular to the plane.

(a) Explain why

$$(\underset{\sim}{r} - \underset{\sim}{r}_0) \cdot \underset{\sim}{n} = 0.$$

(b) Hence, show that the equation of the plane is

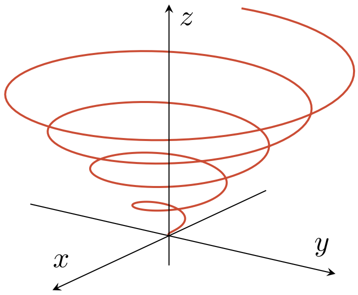
$$ax + by + cz = ax_0 + by_0 + cz_0.$$

(c) Find the equation of the plane that passes through $(2, -3, 1)$ and is perpendicular to the vector $-3\,\underset{\sim}{i} + 2\,\underset{\sim}{j} - \underset{\sim}{k}$.

(d) What is the shortest possible distance between any point on the plane $ax + by + cz + d = 0$ and the origin?

Question 15 The diagrams below show sketches of eight parametrically defined curves.

(i)



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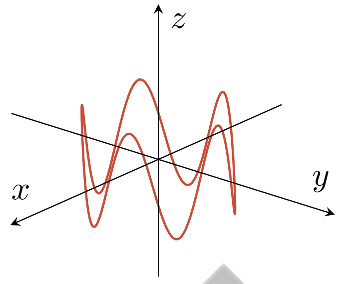

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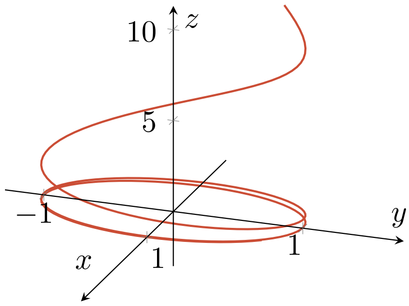

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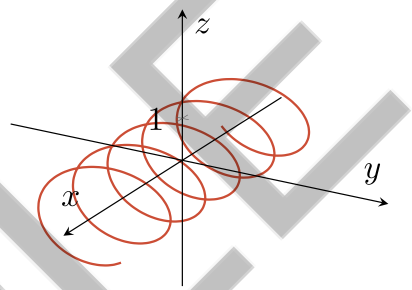

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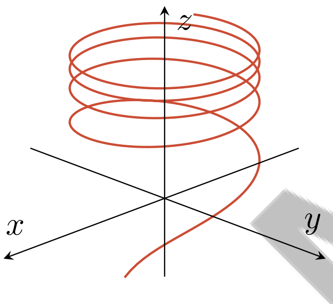

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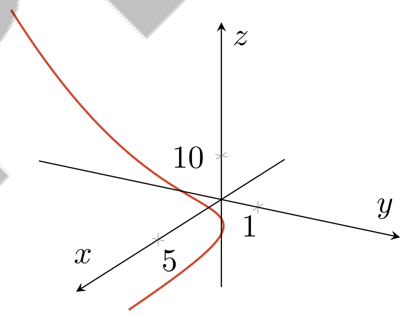

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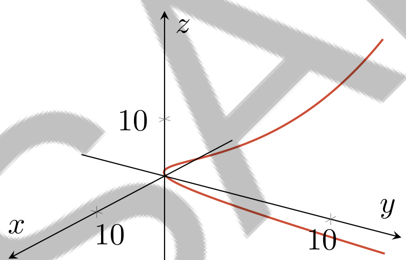

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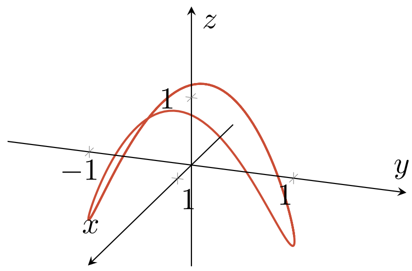


Match the curves to the appropriate set of parametrisations below.

(a) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (e^{-0.5t})\underset{\sim}{k}$

(b) $\underset{\sim}{r}(t) = (t\cos 5t)\underset{\sim}{i} + (t\sin 5t)\underset{\sim}{j} + (t)\underset{\sim}{k}$

(c) $\underset{\sim}{r}(t) = (t)\underset{\sim}{i} + (t^2)\underset{\sim}{j} + (t^3)\underset{\sim}{k}$

(d) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (\cos 2t)\underset{\sim}{k}$

(e) $\underset{\sim}{r}(t) = \cos(t)\underset{\sim}{i} + \sin(t)\underset{\sim}{j} + (\ln t)\underset{\sim}{k}$

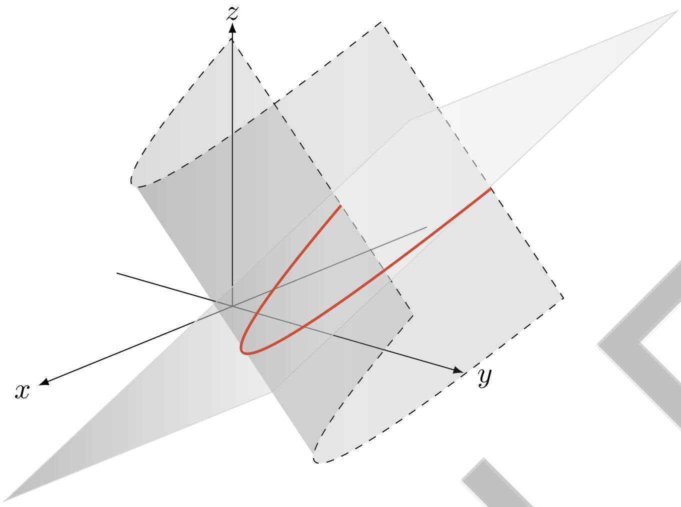
(f) $\underset{\sim}{r}(t) = (e^t)\underset{\sim}{i} + (t)\underset{\sim}{j} + (t^2)\underset{\sim}{k}$

(g) $\underset{\sim}{r}(t) = (t)\underset{\sim}{i} + (\cos 6t)\,\underset{\sim}{j} + (\sin 6t)\underset{\sim}{k}$

(h) $\underset{\sim}{r}(t) = (\cos t)\underset{\sim}{i} + (\sin t)\underset{\sim}{j} + (\sin 5t)\underset{\sim}{k}$

Question 16 [Guided question to find the intersection of two surfaces]

The diagram below shows a parabolic cylinder $z = x^2 - y$ and a plane $z = y - 1$.



(a) Solve the surfaces simultaneously to show that $y = \frac{1}{2}\left(x^2 + 1\right)$. What is the geometric significance of this result?

(b) Set $x = t$ for some $t \in \mathbb{R}$ and hence show that the intersection of the two surfaces has parametric representation

$$\underset{\sim}{r}(t) = (t)\,\underset{\sim}{i} + \frac{1}{2}\left(t^2 + 1\right)\underset{\sim}{j} + \frac{1}{2}\left(t^2 - 1\right)\underset{\sim}{k}.$$

(c) For the $\underset{\sim}{k}$-component above, the surface $z = y - 1$ was used. Is it incorrect to instead use the other surface $z = x^2 - y$?

(d) Suppose that the condition $z \leq 4$ were introduced. Find a corresponding restriction for t and hence find the endpoints of the curve of intersection.

Exercise 4G

Integration by Parts

Fundamentals

Fundamentals 1
Complete the following formula for integration by parts.

$$\int uv' \, dx =$$

Fundamentals 2
Complete the following formula for integration by parts.

$$\int_a^b uv' \, dx =$$

Fundamentals 3

(a) When integrating an isolated function using integration by parts, it is often fruitful to set $v' =$ ___ to introduce an x term.

(b) When selecting what goes into the v' term, it is important to ensure that it will be easy to i________.

Question 1 Find the following using integration by parts.

(a) $\int xe^x \, dx$ (b) $\int x \ln x \, dx$ (c) $\int \sqrt{x} \ln x \, dx$

(d) $\int x \sin x \, dx$ (e) $\int x \sec^2 x \, dx$ (f) $\int x \sin^{-1}(x) \, dx$

(g) $\int x \tan^{-1}(x) \, dx$ (h) $\int \frac{\ln x}{x^2} \, dx$ (i) $\int x \sin x \cos x \, dx$

Question 2 [Integrating isolated functions]

Find the following.

(a) $\int \ln x \, dx$ (b) $\int \sin^{-1}(x) \, dx$ (c) $\int \tan^{-1}(x) \, dx$

(d) $\int \ln(x^2 + 1) \, dx$ (e) $\int e^{\sqrt{x}} \, dx$ (f) $\int \sin(\ln x) \, dx$

Question 3 [Two applications of integration by parts needed]

Find the following.

(a) $\int e^x \sin x \, dx$ (b) $\int x^2 \sin x \, dx$ (c) $\int (\ln x)^2 \, dx$

Question 4 [Prioritising the v' term]

Find the following.

(a) $\int x^5\sqrt{1+x^3}\,dx$ (b) $\int \frac{x^7}{\sqrt{1+x^4}}\,dx$ (c) $\int x^3\cos\left(x^2\right)\,dx$

(d) $\int x^3 e^{x^2}\,dx$ (e) $\int e^{6x}\sin\left(e^{3x}\right)\,dx$ (f) $\int x^3\sqrt{1-x^2}\,dx$

Question 5 Find $\int \sec^3 x\,dx$ using integration by parts.

Question 6 [Definite integrals using integration by parts]

Evaluate the following.

(a) $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ (b) $\int_1^{e} x\ln x\,dx$ (c) $\int_0^1 \tan^{-1}(x)\,dx$

(d) $\int_0^{\frac{\pi}{2}} e^{-x}\cos x\,dx$ (e) $\int_0^{\frac{\pi}{4}} x\tan^2 x\,dx$ (f) $\int_0^1 x^3\tan^{-1}(x)\,dx$

(g) $\int_0^{\infty} xe^{-x}\,dx$ (h) $\int_0^1 x^3 e^{-x^2}\,dx$ (i) $\int_0^{\infty} \frac{\ln(1+e^x)}{e^x}\,dx$

Challenge Problems

Problem 1 Find the following.

(a) $\int \frac{\sqrt{4-x^2}}{x^2}\,dx$ (b) $\int \sqrt{1-x^2}\,dx$ (c) $\int \frac{\ln x}{(1+\ln x)^2}\,dx$

(d) $\int \ln\left(x+\sqrt{x^2-a^2}\right)\,dx$ (e) $\int \frac{\sin^{-1} x}{\sqrt{1+x}}\,dx$ (f) $\int \frac{\tan^{-1}\sqrt{x}}{\sqrt{1+x}}\,dx$

Problem 2 [Application to the Laplace transform]

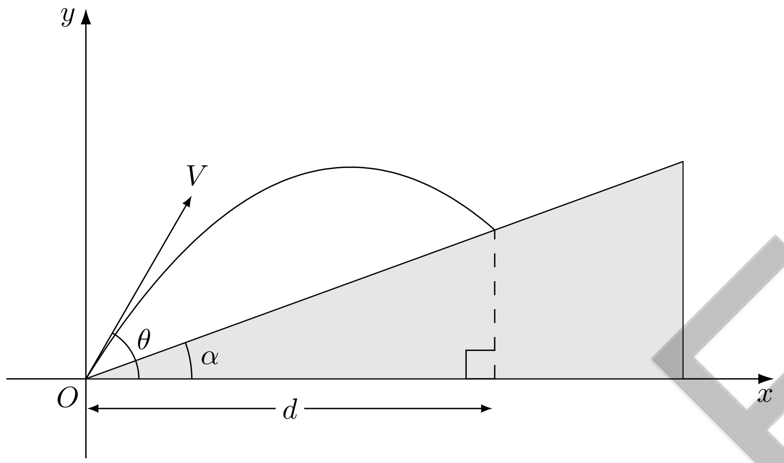
The *Laplace transform* is an advanced technique used to solve differential equations, usually taught in universities. It is an operation on $f(t)$ defined as

$$\mathcal{L}(f(t)) = \int_0^{\infty} e^{-st} f(t)\,dt.$$

The output is a function in terms of s.

(a) Show that $\mathcal{L}(t) = \frac{1}{s^2}$. (b) Show that $\mathcal{L}(\cos at) = \frac{s}{s^2+a^2}$.

Question 7 A particle is launched from the base O of a plane inclined at an angle of α from the horizontal plane.



Initially, the particle has a speed of $V\,\mathrm{m\,s^{-1}}$ and an angle of inclination of θ. You may assume the standard equations of motion in terms of time.

(a) Show that the equation of the trajectory is $y = x\tan\theta - \dfrac{gx^2}{2V^2}\sec^2\theta$.

(b) Show that when the particle hits the ramp, it has travelled a horizontal distance of

$$d = \frac{2V^2\cos\theta\sin(\theta-\alpha)}{g\cos\alpha}.$$

(c) Hence, show that the range of the particle up the inclined plane is

$$R = \frac{2V^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}.$$

(d) Prove that the range R up the ramp is maximised when the angle of projection is halfway between the vertical and the angle of the plane.

(e) Let T be the time of flight when this occurs. Show that $R = \dfrac{1}{2}gT^2$.

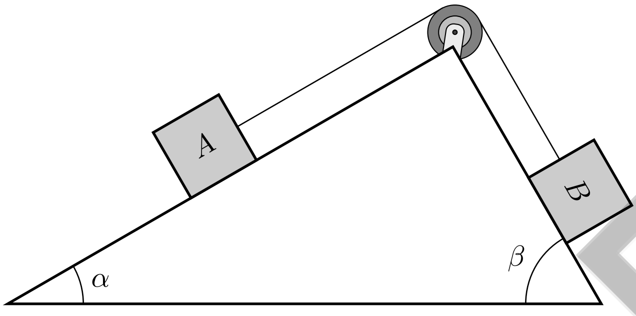
Question 8 A particle is projected from the origin with initial speed $V\,\mathrm{m\,s^{-1}}$ and initial angle θ. The particle passes through the point $P(p, q)$, and has a horizontal range of R.

Show that

$$\tan\theta = \frac{qR}{p(R-p)}.$$

Question 6 [Generalised double smooth ramp problem]

Two objects A and B with masses of $m_1$kg and $m_2$kg respectively are connected by a light inextensible string that runs through a smooth pulley. The objects lean on a double-sided smooth ramp inclined at angles of α and β from the horizontal, as shown below.



The system moves so that particle A slides down the ramp whilst particle B slides up the ramp.

(a) Show that

$$a = g\left(\frac{m_1 \sin\alpha - m_2 \sin\beta}{m_1 + m_2}\right).$$

(b) Hence, show that

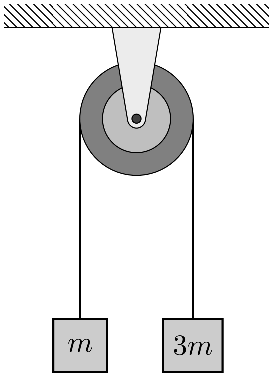
$$T = g\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\sin\alpha + \sin\beta).$$

(c) Show that if

$$\frac{\sin\alpha}{\sin\beta} = \frac{m_2}{m_1},$$

then the system will remain at static equilibrium.

Question 9 The following diagram shows two objects with masses of m kg and $3m$ kg on either end of a light inextensible string that passes through a smooth pulley. Both particles are released from rest simultaneously.



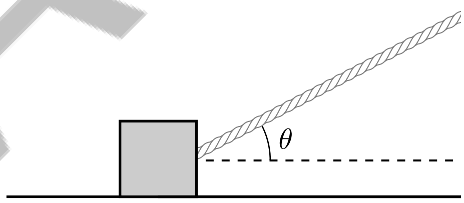
Let a be the acceleration of the heavier particle in the downwards direction. Let g be the acceleration due to gravity.

(a) Show that $a = \frac{g}{2}$.

(b) Hence, show that after 4 seconds, the heavier object travels $4g$ metres and has speed $2g\,\mathrm{m\,s^{-1}}$.

Question 10 [Minimal force problem on a flat surface]

An object of mass m rests on the surface of a table. It is attached to a rope inclined at an angle of θ from the horizontal that pulls it to the right.



The object experiences a friction force $F = \mu N$ that resists the motion of the object.

(a) Resolve forces in the vertical and horizontal directions.

(b) Hence, show that the amount of tension needed to overcome friction is

$$T = \frac{\mu m g}{\cos\theta + \mu \sin\theta}.$$